Martingale-Envelopes: Theory and Applications

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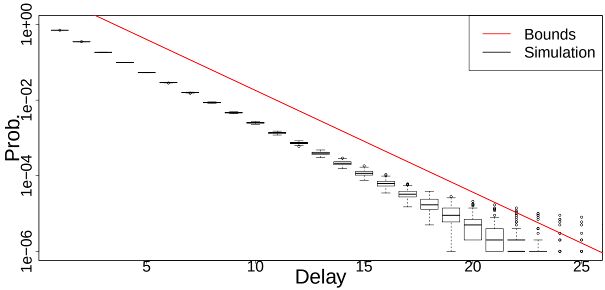
Introduction

The performance bounds of SNC are loose:



Crucial point: Estimate the *supremum* of a stochastic process, i.e.

$$\mathbb{P}(\sup_n X_n \geq \sigma) \leq \sum_n \mathbb{P}\left(X_n \geq \sigma\right) \quad \text{“Boole's inequality”.}$$

Does not account for dependencies/correlation!

Introduction

For a single random variable X

$$\mathbb{P}(X \geq \sigma) \leq \mathbb{E}[X]\sigma^{-1} \quad \text{“Markov inequality”}.$$

Extension to stochastic processes?
Supermartingales:

$$\mathbb{P}(\sup_n X_n \geq \sigma) \leq \mathbb{E}[X_0]\sigma^{-1} \quad \text{“Doob inequality”}.$$

Supermartingales

Definition

A *supermartingale* is a process X_n such that for each $n \in \mathbb{N}$

$$\mathbb{E}[X_{n+1} - X_n \mid X_1, \ldots, X_n] \leq 0 \ .$$

The *expected increment* is negative.
Analogy: In a queueing system,

$$\text{average rate} \leq \text{capacity}, \quad \text{“Loynes condition”},$$

expected *change of the buffer content* is negative as well.

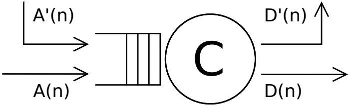
Supermartingales

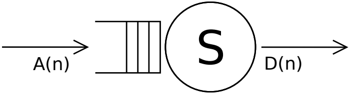
Idea: Assign to a queueing system a certain supermartingale M_n ("Martingale-Envelope").

- *Multiplexing* results in multiplication of the supermartingales.
- *Scheduling* results in switching between the supermartingales.
- The resulting performance bounds become reasonably tight.

The Model

Discrete time $n \in \mathbb{N}$,   stationary arrival processes





- through- and crossflow
- constant capacity

- only throughflow
- stochastic service process

Performance metrics:

- Backlog: $Q := \sup_{n\in\mathbb{N}}(A(n) - Cn)$
- Delay: $W(n) := \inf\{k \in \mathbb{N} \mid A(n-k) \leq D(n)\}$

Martingale-Envelope

Definition

For $\theta > 0$ and h monotonically increasing, the flow A admits a (h, θ, C)-martingale-envelope if

$$M(n) := h(a_n)e^{\theta(A(n)-Cn)}$$

is a supermartingale.

- C is the allocated capacity
- θ and h capture the correlation structure of A

Estimation of the supremum

Define the threshold

$$\tau_{A,C} := \inf\{x > C \mid \mathbb{P}(a_k \in [x, \infty)) > 0\}.$$

as the smallest instantaneous arrival larger than C.
With a variant of Doob's inequality,

Backlog:

$$\mathbb{P}(Q \geq \sigma) \leq \frac{\mathbb{E}[h(a_0)]}{h(\tau_{A,C})} e^{-\theta\sigma}$$

Delay:

$$\mathbb{P}(W(n) \geq k) \leq \frac{\mathbb{E}[h(a_0)]}{h(\tau_{(A,C)})} e^{-\theta Ck} .$$

Multiplexing

Two independent flows A_1 and A_2 admitting martingale-envelopes with (h_1, θ, C_1) and (h_2, θ, C_2).

Define

$$h_1 \otimes h_2(a) = \inf_{0 \le b \le a} h_1(b) h_2(a-b) \ ,$$

"(min, $\times$)-convolution".

- $h_1 \otimes h_2$ is the smallest function with $h_1 \otimes h_2(a+b) \le h_1(a) h_2(b)$
- if h_1,h_2 monotonic so is $h_1 \otimes h_2$

Multiplexing

The aggregate flow $A_1 + A_2$ admits a martingale-envelope with parameters

$$(h_1 \otimes h_2, \theta, C_1 + C_2) \ .$$

Proof:
$M_1(n)$ and $M_2(n)$ corr. supermartingales, $M_1(n)M_2(n)$ is a supermartingale as well!

Scheduling

- Only interested in performance of flow A_1
- Challenge: Plug in the service process into the martingale calculus!

Obervation: For a “swiching time“ $l \in \mathbb{N}$, the process

$$\tilde{M}(n) = \begin{cases} M_2(n) & n \leq l \\ M_2(n)M_1(n) & n \geq l \end{cases}$$

is a supermartingale!
Sample path bound:

$$\mathbb{P}\left(\sup_{0 \leq m < n-l} \{A_1(m, n-l) + A_2(m,n) - C(n-m)\} \geq \sigma\right) \leq \frac{\mathbb{E}[h_1(a_0)]\mathbb{E}[h_2(a_0)]}{h_1 \otimes h_2(\tau_{A_1+A_2, C_1+C_2})} e^{-\theta(\sigma + C_1 l)} .$$

Scheduling

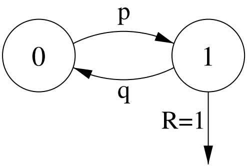
For each scheduling policy plug in an appropriate switching time l!
For the delay $\mathbb{P}(W(n) \geq k)$:

$$\begin{aligned} \text{FIFO: } l &= 0 && \leq \kappa e^{-\theta C k} \\ \text{SP: } l &= k && \leq \kappa e^{-\theta C_1 k} \\ \text{EDF: } l &= y := d_1 - d_2 && \leq \kappa e^{-\theta(Ck - C_2 \min(k,y))} \,, \end{aligned}$$

where

$$\kappa := \frac{\mathbb{E}[h_1(a_0)]\mathbb{E}[h_2(a_0)]}{h_1 \otimes h_2(\tau_{A_1+A_2,C_1+C_2})} \,.$$

Application: on-off-processes



- two-state Markov chain a_n,
  $A(n) = \sum_{k=1}^{n} a_k$.
- stationary distribution $\pi = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$

Transition matrix:

$$T := \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \rightsquigarrow T_\theta := \begin{pmatrix} 1-p & pe^\theta \\ q & (1-q)\,e^\theta \end{pmatrix},$$

$\lambda(\theta)$ max. pos. eigenvalue, (v_0, v_1) corr. pos. eigenvector

For a specific value of θ:

$$M(n) = v_{a_n} e^{\theta(A(n)-Cn)} \text{ is a martingale.}$$

$\Rightarrow A$ admits a (v, θ, C)-martingale-envelope!

Calculating the constant

Multiplexing N independent on-off-sources A_i.
The constant:

$$\kappa = \frac{\mathbb{E}[h(a_0)]^N}{h^{\otimes N}(\tau_{\sum A_i, NC})} = \frac{(\pi_0 v_0 + \pi_1 v_1)^N}{v_0^{N - \lceil NC \rceil} v_1^{\lceil NC \rceil}}$$

In the case of $p < 1 - q$ ("bursty traffic"), $v_0 < v_1$ and thus

$$\kappa \leq \left(\frac{\pi_0 v_0 + \pi_1 v_1}{v_0^{1-C} v_1^C} \right)^N$$

"Multiplexing Gain": Exponential decay in the leading constant!

Applying the martingale bound

For the aggregate flow:

$$\mathbb{P}\left(Q \geq \sigma\right) \leq \kappa e^{-\theta\sigma}$$
$$\mathbb{P}\left(W(n) \geq k\right) \leq \kappa e^{-\theta NCk}$$

and for the single flow comprising $N_1 < N$ subflows:

FIFO: $\mathbb{P}\left(W(n) \geq k\right) \leq \kappa e^{-\theta NCk}$

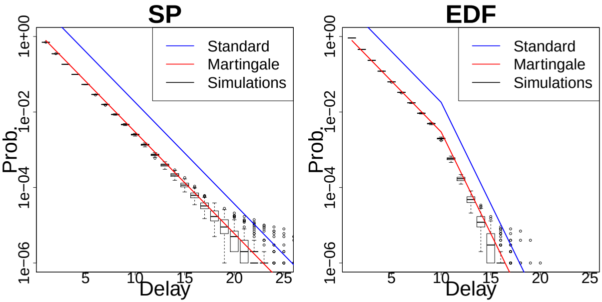
SP: $\mathbb{P}\left(W(n) \geq k\right) \leq \kappa e^{-\theta N_1 C_1 k}$

EDF: $\mathbb{P}\left(W(n) \geq k\right) \leq \kappa e^{-\theta(NCk-(N-N_1)C\min(k,d_1-d_2))}$

Simulations

Consider the on-off-process:

- $N_1 = 10,\ N = 20$
- $p = 0.1,\ q = 0.5$
- $\rho = 75\% \Rightarrow C = 0.22$
- $y = d_1 - d_2 = 9$ for EDF



Extensions/Challenges

Martingale-Envelopes can be constructed for:

- other Markov driven processes
  - incl. i.i.d. processes
- p-order autoregressive processes
  - explicit solutions!

Challenges:

- broader class of arrival models (long-range dependent ...)
- multi-hop scenarios

Conclusions

- Characterize the queueing-system by a supermartingale
- Multiplexing and Scheduling result in multiplication and switching of the martingales
- Apply Doob's maximal inequality

$\Rightarrow$ The bounds become reasonably tight!