ABSTRACT
ACM Sigcomm 2006 published a paper [26] which was perceived to unify the deterministic and stochastic branches of the network calculus (abbreviated throughout as DNC and SNC) [39]. Unfortunately, this seemingly fundamental unification—which has raised the hope of a straightforward transfer of all results from DNC to SNC—is invalid. To substantiate this claim, we demonstrate that for the class of stationary and ergodic processes, which is prevalent in traffic modelling, the probabilistic arrival model from [26] is quasi-deterministic, i.e., the underlying probabilities are either zero or one. Thus, the probabilistic framework from [26] is unable to account for statistical multiplexing gain, which is in fact the raison d’être of packet-switched networks. Other previous formulations of SNC can capture statistical multiplexing gain, yet require additional assumptions [12, 22] or are more involved [14, 9, 28], and do not allow for a straightforward transfer of results from DNC. So, in essence, there is no free lunch in this endeavor.

Our intention in this paper is to go beyond presenting a negative result by providing a comprehensive perspective on network calculus. To that end, we attempt to illustrate the fundamental concepts and features of network calculus in a systematic way, and also to rigorously clarify some key facts as well as misconceptions. We touch in particular on the relationship between linear systems, classical queueing theory, and network calculus, and on the lingering issue of tightness of network calculus bounds. We give a rigorous result illustrating that the statistical multiplexing gain scales as $\Omega(\sqrt{N})$, as long as some small violations of system performance constraints are tolerable. This demonstrates that the network calculus can capture actual system behavior tightly when applied carefully. Thus, we positively conclude that it still holds promise as a valuable systematic methodology for the performance analysis of computer and communication systems, though the unification of DNC and SNC remains an open, yet quite elusive task.
raison d’être of SNC was to additionally capture statistical multiplexing gain when some violations of the deterministic bounds are tolerable. This feature enables a much more efficient dimensioning of resource sharing systems, such as packet-switched networks, and continues to play a pivotal role in the evolution of SNC.

The promise of the combined branches of network calculus is to jointly overcome the technical barriers of queueing networks on all fronts. Achieving this rather daunting task is enabled by two key features:

- **Scheduling abstraction.** At a single queue with multiplexed arrival flows, the specific properties of many scheduling algorithms, and also of many arrival classes, can be abstracted away by suitably constructing the so-called service processes (technical details are deferred to Section 3).

- **Convolution-form networks.** The service processes from single queues can be convolved across a network of queues, and thus a multi-node network analysis can be drastically simplified by reducing it to a single-node analysis.

Equipped with these two features, the network calculus can analyze many scheduling algorithms and arrival classes, over a multi-node network, in a uniform manner. That means that the Poisson model, in particular, plays no special role anymore in facilitating the analytical tractability of a whole network. Compared to classical queueing theory, which separately analyzes various combinations of arrivals and scheduling, network calculus conceivably offers a much more simplified and uniform framework. For this reason, the network calculus has been applied in many recent areas such as IntServ [6], switched Ethernet [44], systems-on-chip [10], avionic networks [41], the smart grid [48], etc.

This versatile applicability, however, is only possible at the expense of providing bounds on performance metrics. The bounds are a manifestation of resorting to inequalities, whenever exact derivations become intractable. The tightness of the bounds is certainly a major concern, since loose bounds may be more misleading than wrongly fitted Poisson models. The tightness issue has several dimensions depending on the nature of the bounds (deterministic or probabilistic) or the number of flows/queues. Deterministic bounds are generally tight for single queues ([5], p. 27), but can be very loose in some queueing networks with arbitrary multiplexing [43]. Moreover, the deterministic bounds can be very inefficient for network dimensioning when some violation probabilities are tolerable (e.g., running IntServ for many flows could result in very low network utilization). Probabilistic bounds are generally asymptotically tight (in terms of scaling laws in the number of queues) [8, 34], whereas numerical tightness ranges from reasonable [13] to quite loose [34], depending on the arrival model.

In this paper we touch on the lingering issue of tightness, as part of a broader perspective on network calculus. Concretely, we address the asymptotic tightness of probabilistic bounds, in the number of flows $N$, and demonstrate that such bounds improve upon corresponding deterministic bounds by a factor of $\Omega(\sqrt{N})$. That means that, e.g., implementing a probabilistic extension of IntServ could significantly increase the network utilization. Our result not only rigorously reveals the magnitude of the statistical multiplexing gain achieved with SNC, but clearly highlights the fundamental advantage of SNC over DNC.

Our broader goal is to deliver an intuitive and yet comprehensive perspective of the two core concepts in network calculus, i.e., service and envelope processes, by focusing on subtleties and raising awareness of inherent pitfalls. Along the discussion we attempt to make a suggestive statement that there is no free lunch in the framework of the network calculus, yet it brings good value as a companion/alternative to the classical queueing theory. This perspective is motivated by a large effort in the literature to develop SNC formulations which reproduce in particular the “convolution-form networks” feature from DNC. Arguably the simplest of such formulations has appeared in a Sigcomm 2006 paper [26], and has since raised the hope that DNC results can be transferred into SNC in a straightforward manner. Unfortunately, the formulation from [26] is based on a quasi-deterministic arrival model\(^1\), which roughly means that the proposed SNC cannot capture statistical multiplexing gain. We believe that exposing this pitfall, through a rigorous analysis, is essential to the comprehensive understanding of SNC arrival models.

After introducing notations, the rest of the paper is structured as follows. In Section 3 we provide a comprehensive perspective on service processes by making a multilateral analogy of network calculus with linear systems and classical queueing theory. In Section 4 we present representative envelope processes and elaborate on the quasi-deterministic aspects of the one from [26]. In Section 5 we lead together envelope and service processes in order to shed some light on the often raised concern about the tightness of network calculus bounds. We conclude the paper in Section 6.

2. NOTATIONS

The time model is discrete starting from zero. The time indices are denoted by the symbols $i,k,n$. The cumulative arrivals and departures at/from a (queueing) node up to time $n$ are denoted by non-decreasing processes $A(n)$ and $D(n)$. The doubly-indexed extensions are $A(n,k)=A(n)-A(k)$ and $D(k,n)=D(n)-D(k)$. The associated instantaneous arrival and departure processes are $a_n=A(n-1,n)$ and $d_n=D(n-1,n)$, respectively; by convention, $a_0=d_0=A(0)=D(0)=0$. The vector representations are $A=(A(0),A(1),\ldots)$ and $a=(a_0,a_1,\ldots)$ for the arrivals, and $D=(D(0),D(1),\ldots)$ and $d=(d_0,d_1,\ldots)$ for the departures. These processes have primarily a spatial interpretation, e.g., $a_n$ quantifies the number of data units (referred to as bits throughout) arrived at time $n$; with abuse of notation, $a$ and $d$ will also have a temporal meaning to be made locally clear.

The sets of natural, integer, real, and positive real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}_+$, respectively; their restriction to non-zero numbers are denoted by $\mathbb{N}^*$, $\mathbb{Z}^*$, $\mathbb{R}^*$, and $\mathbb{R}_+^*$. The integer part of a number $x \in \mathbb{R}$ is denoted by $\lfloor x \rfloor$. For $x \in \mathbb{R}$, the positive part is denoted by $x_+ = \max(x, 0)$. For some boolean expression $E$, the indicator function is denoted by $\text{1}_E$ and takes the values 1 or 0 depending whether $E$ is true or false, respectively.

\(^1\)The authors of [9] also point out the quasi-determinism issue in [26], but for a service model and without proof; as a further side remark, we have ourselves experienced a similar quasi-determinism pitfall in SNC [42].
3. SERVICE PROCESSES

Network calculus operates by reducing a ‘complex’ non-linear (queueing) system into a ‘somewhat looking’ linear system. Because the reduced system is often analytically tractable—linearity conceivably implies simplicity—network calculus is regarded as an attractive approach to analyze complex queueing systems. In this section we elaborate on this key reduction operation by exploring conceptual similarities with the more traditional linear system and queueing theories, as well as on its main diverging point from the two. The final goal is to highlight the emergence of the concept of a service process, which is instrumental for abstracting away some of the technical challenges characteristic of non-linear systems.

The ‘complex’ system is a node, or a network of nodes, in which bits arrive and depart according to various factors such as probability distributions for arrival processes, scheduling, routing, etc. A fundamental networking and queueing rivals have the same size of one bit. The construction of the corresponding outputsignal v (also called impulse-signal) and its corresponding output signal v (also called impulse-response)

\[ u_n = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}, \quad v = T(u). \]  

The impulse signal u is (technically) motivated by the convolution property \( a_n = \sum_{k=-\infty}^{\infty} a_k u_{n-k} \forall a_n. \) With these assumptions, it can be shown that for any input signal \( a, \) the corresponding output signal \( d \) can be completely determined by the following convolution

\[ d_n = \sum_{k=-\infty}^{\infty} a_k u_{n-k} \forall n \in \mathbb{Z}. \]  

This result is central in linear system theory, as it simply solves for \( T. \) This solution for the SI problem, however, relies on the strong LTI assumption from Eq. (1).

3.1 Linear System Theory

The SI problem has a direct correspondent in linear system theory [31]. Assume that \( T \) is linear and time-invariant (LTI), i.e.,

\[ \begin{align*}
T(c_1 a_1 + c_2 a_2) &= c_1 T(a_1) + c_2 T(a_2) \\
T(a_{(-k)}) &= T(a)_{(-k)}
\end{align*} \]  

for all signals \( a, a_1, a_2, \) scalars \( c_1, c_2, \) and integers \( k. \) Here we tacitly assume ‘signal’ interpretations of the input and output sequences, and also their extension to doubly infinite sequences such that the shifted version \( a_{(-k)} \) of \( a, \) i.e., \( a_{(-k)} := a_{n-k} \forall n, k \in \mathbb{Z}, \) is well defined. Define the Kronecker input signal \( u \) (also called impulse signal) and its corresponding output signal \( v \) (also called impulse-response)
These equations are fundamental elementary identities in queueing theory.

The operations from Eqs. (4) and (5) on input sequences resemble much with the convolution operation from Eq. (3), except for the underlying algebra. Concretely, while Eq. (3) is formed according to the traditional convolution involving the addition of products, Eqs. (4) and (5) are formed by minimizing and maximizing, respectively, sums. For this reason, it is said that the operator $T$ operates in a $(\min, +)$ algebra in Eq. (4), and in a $(\max, +)$ algebra in Eq. (5).

We have thus presented another complete characterization of $T$. What makes this second solution partial as well is that the considered queueing system is noiseless, i.e., it assumes a constant-rate server, no scheduling, etc.

### 3.3 Emergence of the Service Process Concept

We now combine the ideas from the previous two subsections in order to present a much more general solution for the underlying LTI assumption from Eq. (7). In the following we present two queueing examples which show that, in spite of the fact that many queueing systems are generally not $(\min, +)$ linear, the unknown operator $T$, and thus a solution for the SI problem, can be constructed in great generality. Since there is no free lunch, this promising increase in generality is only possible at an inevitable price: sacrificing exactness, or, more concretely, replacing the equality from Eq. (6) by an inequality, i.e., $D \geq T(A)$.

#### 3.3.1 Example 1

Consider the (noiseless) queueing system from Figure 1.(a) (see next page) and recall the relationship from Eq. (4) between departures and arrivals. Fitting this relationship with Eq. (10) yields $S(n) = nC \forall n \in \mathbb{N}$. As mentioned, $S$ also has the physical interpretation of the cumulative output from the queue when the input is the impulse function $\delta$ from Eq. (8).

This example, although repetitive, is meant to illustrate the (almost perfect) analogy in the arguments used in Sections 3.1 and 3.2. Concretely, it is apparent that there is a match between the construction of $S$ from 1) the $(\min, +)$ linear system theory (as in Eqs. (9) and (10)), and 2) elementary queueing properties (as in Eq. (4)). The missing element for a perfect analogy is that the queueing system is $(\min, +)$ LTI under an artificial interpretation of the required ‘plus’ property $T(c + A) = c + T(A)$ from Eq. (7): the addition of scalars $c$ occurs, both in the input and output, before the queueing system actually starts. This is clearly an inconvenient physical system interpretation, but it enables the view of a constant-rate queueing server as a $(\min, +)$ LTI system. As a side remark, the physically more meaningful interpretation of the ‘plus’ property as a burst of $c$ bits at time zero has the negative consequence that any system of practical interest, in particular the constant-rate work-conserving server, would be $(\min, +)$ non-linear. This can be seen by a result in ([5], Proposition 8.3.1) on a system with non-empty initial buffer clearly exhibiting a $(\min, +)$ non-linear behavior.

#### 3.3.2 Example 2

Making an analogy between $(\min, +)$ LTI systems and elementary queueing properties is even more compounded for the FIFO queueing system from Figure 1.(b), which includes noise in the form of cross-traffic. The reason is that this queueing system is not anymore $(\min, +)$ linear, even under the artificial interpretation of the ‘plus’ operation. First, the ‘min’ property $T(\min \{A_1, A_2\}) = \min \{T(A_1), T(A_2)\}$
fails: a quick example is $C = 3$, $A_1(t) = (0, 1, 3, 4)$, $A_2(t) = (0, 3, 4, 6)$, and $A_3(t) = (0, 1, 7, 10)$. Second, the time invariance property fails as well: a quick example is $C = 1$, $A = (0, 1, \ldots)$, $A = (0, 1, 3, \ldots)$, and the right shift $k = 2$. Because the queueing system is not $(\min, +)$ LTI, one cannot follow the construction of the impulse-response sequence $S$ in order to exactly characterize the queueing system as in Eq. (10).

At this apparent impasse, the network calculus slightly diverges from LTI systems and queueing theory. The key idea is to transform the non-linear queueing system into a ‘somewhat looking’ linear system. The actual transformation occurs by directly constructing a ‘somewhat analogous’ impulse-response $S$ (a.k.a. service process) [1] satisfying

$$D \geq A * S \forall A.$$  (11)

Therefore, instead of exactly characterizing the system as in Eq. (6), the network calculus makes the crucial concession of inexactly characterizing the system by resorting to an inequality, as in Eq. (11). For the FIFO multiplexing example, one choice of a service process $S$ is the bivariate random process $S_2(k, n) = \left[C(n-k) - A_2(k, n-i)\right]_+ 1_{(n-i>3)}$, for some $i \geq 0$ [18], which satisfies

$$D(n) \geq \min \{ A(k) + S_k(n) \} \forall A, n \in \mathbb{N}.$$  (12)

Except for the inequality, this characterization resembles much with both Eq. (10) (from $(\min, +)$ LTI systems theory) and Eq. (4) (from queueing theory). Note, however, the non-trivial expression of $S_2(k, n)$ stemming from non-trivial characteristics of FIFO multiplexing. In particular, the bivariate form is due to the lack of time invariance.

The FIFO multiplexing example reflects a fundamental tradeoff in network calculus. On one hand, as queueing systems are generally neither linear nor time invariant, network calculus resorts to inequalities for their characterization, as in Eqs. (11) or (12). On the other hand, the service processes $S$ ought to be reasonably concise and also provide tight bounds in Eq. (11); otherwise, they may render a cumbersome analysis (see, e.g., [33] for the above choice of $S$) or simply arbitrarily loose bounds. Constructing such ‘nice’ service processes for existing scheduling algorithms is a key challenge in network calculus; see [36] for some state-of-the-art examples of service processes concerning Δ-scheduling, which generalizes FIFO, static priority, and earliest deadline first (EDF).

Nevertheless, the underlying methodology of constructing service processes to abstract away the details of scheduling algorithms, in queueing scenarios with many flows, renders two central features of network calculus: scheduling abstraction and convolution-form networks. With the former, many classes of scheduling policies and arrival processes are amenable to a uniform analysis. In other words, once service processes are suitably constructed, the network calculus analysis does not conceptually differentiate between, e.g., FIFO and EDF policies, or Poisson and Markov arrival processes, for the purpose of computing per-flow (or per-class) performance metrics. With the latter feature, the multi-node queueing analysis is drastically simplified. Concretely, once service processes $S'$ are constructed at each node along a network path, the entire network analysis can be reduced to the analysis of a single-node, which is characterized by the following service process

$$S = S^1 * S^2 * \ldots * S^n,$$  (13)

i.e., the convolution of the service processes along the network path. What makes this reduction particularly appealing is that the multi-node performance bounds obtained in this manner are asymptotically tight in the number of nodes (see, e.g., [14, 22]).

In conclusion, network calculus provides a methodology to solve the SI problem by transforming a non-linear system (subject to various arrivals, scheduling, or multi-node) into a ‘somewhat looking’ linear system which is amenable to a quite straightforward analysis. The key challenge is the transformation itself, i.e., the construction of ‘nice’ service processes.

Most of the interpretations on network calculus illustrated in this section appear in the literature in isolation: for the analogy with linear systems see [19, 12, 5], for the analogy with queueing theory see [27], for a discussion on the non-linearity of FIFO systems see [35]. For a comprehensive survey of service processes we refer to [23]. Our contribution herein was to present a comprehensive perspective on the emergence and central role of service processes in network calculus, by weaving together linear systems, queueing theory, and network calculus.

4. ENVELOPE PROCESSES

We now shift the discussion to the other fundamental concept in network calculus: envelope processes. While their role is to model a very broad class of arrival processes, achieving this generality comes at the price of sacrificing exactness in the arrivals’ representation. The goal of this section is to highlight the key aspects of envelope processes; it is not meant to provide a review of the types of envelopes, for which we refer to [37].

A (cumulative) arrival process $A(n)$ is typically described by either a complementary cumulative distribution function (CCDF) or a moment generation function (MGF), i.e.,

$$F_{A(n)}(\sigma) := P \left( A(n) > \sigma \right), \quad M_{A(n)}(\theta) := E \left[ e^{\theta A(n)} \right],$$

respectively, for all $n \in \mathbb{N}, \sigma \in \mathbb{R},$ and $\theta \in \Theta$, where $\Theta$ is some space over $\mathbb{R}$. The two descriptions silently assume that $A(n)$ is a stationary random process, which means that the CCDF is invariant under time shift.

The existence of the MGF is equivalent to an exponentially bounded CCDF, in which case the CCDF uniquely determines the MGF according to the identity relating expectations and tails, i.e., $E[X] = \int_0^\infty p(X > x)dx$ for positive random variable (r.v.) $X$. Conversely, in the case when $\Theta$ is an open interval including zero, the MGF uniquely determines the CCDF according to analytic function theory ([21], p. 274). Throughout we consider arrival processes which have an MGF.
Let us consider the following example of a compound Bernoulli arrival process

\[ A(n) = \sum_{k=1}^{n} X_k \quad \forall n \in \mathbb{N}, \]

where \( X_k \)'s are i.i.d. Bernoulli(\( p \)) r.v.'s taking the values 1 and 0 with probabilities \( p \) and \( 1 - p \), respectively. The simplicity of \( A(n) \) will enable the illustration of some key insights in an intuitive and yet rigorous manner. The corresponding expressions for the CCDF and MGF are

\[
F_{A(n)}(\sigma) = \left\{ \begin{array}{ll}
  1, & \sigma < 0 \\
  \sum_{k=1}^{n} \binom{n}{k} p^k (1 - p)^{n-k}, & \sigma \geq 0
\end{array} \right. ,
\]

\[
M_{A(n)}(\theta) = e^{\theta n},
\]

where \( \theta \) is a rate and \( \theta \in \mathbb{R}^+ \).

Although the process \( A(n) \) is fully described, the network calculus provides more 'flexibility' by offering further modelling alternatives depending on two factors:

1. type of analysis: deterministic (a.k.a. worst-case) or probabilistic. The former seeks to yield statements like "The (queueing) delay is smaller than some number." The latter seeks statements like "The delay is smaller than some number with some probability."

2. tradeoff between accuracy and elegance of the analysis itself.

For instance, in the case of a deterministic analysis, one must resort to deterministic models to (partially) suppress the uncertainty of arrivals. The suppression process occurs by replacing random processes with deterministic functions, i.e., the amount of bits in the system at time \( n \) is defined as

\[ \text{B}_{\text{n}} = A(n) - D(n), \]

i.e., the amount of bits in the system at time \( n \). Recalling the expression for \( D(n) \) from Eq. (4), one may immediately derive Reich's equation:

\[ \text{B}_{\text{n}} = \max_{0 \leq k \leq n} \{ A(k, n) - C(n - k) \} . \]

This equation clearly indicates that the bounding model from Eq. (17) is insufficient to get a non-degenerate bound on the backlog process. That is because a bound on \( A(n) \) does not necessarily induce a bound on \( A(n, k) \), which is what is actually needed. For a quick example, take the arrival process \( A = (0, 0, 20) \) and \( G(n) = 10n \); clearly, \( A(n) \leq G(n) \) but \( A(1, 2) = 20 > 10 = G(1) \). A more intuitive argument is that the backlog process depends on the entire history of the process, and relative to any time point, whereas Eq. (17) only captures the process relative to the origin.

Let us next make an additional stationarity assumption on \( A(n) \). Then, if a deterministic computation of queueing measures was the goal, the model from Eq. (17) would remain insufficient; in fact, the model would only be sufficient for constant-rate arrivals. However, if a probabilistic computation was the goal, then the model from Eq. (17) could become useful. Indeed, a computation of the backlog tail would be for instance

\[ \mathbb{P}(\text{B}_n > \sigma) = \mathbb{P} \left( \max_{0 \leq k \leq n} (A(k, n) - C(n - k)) > \sigma \right) \leq \sum_{k=0}^{n} \mathbb{P}(A(k, n) - C(n - k) > \sigma). \]

The last line follows from Boole's inequality\(^2\). As a side remark, we point out that this apparently loose inequality is

\[^2\text{For some probability events E and F, Boole's inequality is } \mathbb{P}(E \cup F) \leq \mathbb{P}(E) + \mathbb{P}(F).\]
not so bad if the r.v.'s $X_k := A(k, n)$ are rather uncorrelated (e.g., when $A(n)$ is a (discretized) Poisson process), but it is quite loose if they are highly correlated [46]. In such a case one can make use of more sophisticated techniques with refined martingale inequalities (see [12], pp. 339-343). For the purpose of our presentation it is sufficient to adopt the simplified technique with Boole's inequality.

Due to the stationarity assumption, the last line can be continued by replacing $A(k, n)$ with $A(n - k)$, and further by $\mathcal{G}(n - k)$ according to the bounding arrival model from Eq. (17). Note however that $\mathcal{G}(n)$ should be defined as a random process, and the inequality from Eq. (17) should hold a.s. (almost surely, i.e., $\mathbb{P}(A(n) < \mathcal{G}(n)) = 0$). Otherwise, if $\mathcal{G}(n)$ was a non-random function, then the previous derivation would be quasi-deterministic since all the probabilities would evaluate either to 0 or 1.

4.2 Classic Deterministic Arrival Model

The previous pitfall indicates that, in general, it is not sufficient to bound arrivals on all intervals $(0, n)$ but rather on all intervals $(k, n)$. This observation suggests the following bounding arrival model

$$A(k, n) \leq \mathcal{G}(n - k) \quad \forall 0 \leq k \leq n ,$$

(20)

where $\mathcal{G}(n)$ is a non-random function. Network calculus was essentially founded on this arrival model [16].

With this arrival model, the deterministic continuation of Eq. (18) is straightforward:

$$B_n \leq \max_{0 \leq k \leq n} \{ \mathcal{G}(k) - CK \} .$$

(21)

The RHS term can be computed explicitly in $O(1)$ time if $\mathcal{G}(n)$ is a sufficiently ‘nice’ expression, e.g., $\mathcal{G}(n) = rn + b$ where $r$ and $b$ have the meanings of rate, and burst, respectively. Otherwise, if $\mathcal{G}(n)$ is given pointwise, then the RHS term can be computed in $O(n)$ time.

We emphasize that there is no requirement of stationarity on the arrival process $A(n)$. In fact, the regularity constraint from Eq. (20) is satisfied by infinitely many (and possibly unknown) arrival processes, thus illustrating the high modelling potential of Eq. (20). Moreover, despite the apparent tradeoff between modelling potential and accuracy of representation, the derivation from Eq. (21) is actually tight [5, p. 27]. Tightness means that there exists an arrival process which 1) satisfies the arrival bound from Eq. (20), and 2) induces a backlog process which matches with the predicted bound from Eq. (21). Even more remarkably, the tightness of the backlog bound holds even when multiplexing many possibly ‘conspiring’ flows, e.g., producing large bursts at the same time.

To more concretely elaborate on the tightness of the deterministic modelling and analysis from Eqs. (20)-(21), let us consider an aggregate $A(n) = \sum_{j=1}^{\infty} A_j(n)$, where $A_j(n)$'s are compound Bernoulli processes as in Eq. (14). The flows are multiplexed at a server with capacity $C < N$, and there is no statistical independence assumption amongst them. The performance metric of interest is again a bound on the backlog process $B_n$. According to Eq. (21), to get the tightest bound on $B_n$, one must first construct the smallest non-random function satisfying Eq. (20), which is $\mathcal{G}(n) = Nn$. When plugged into Eq. (21), this yields the bound

$$B_n \leq \max_{0 \leq k \leq n} \{ N(n - k) - C(n - k) \} = (N - C)n .$$

Because $N > C$ the bound diverges and clearly becomes useless when $n \to \infty$. We point out, however, that the bound is tight even under a statistical independence assumption on $A_j(n)$’s. Indeed, $\forall n \geq 0$ there exists a positive probability such that $A_j(n) = 0 \forall j$, i.e., there exists a sample-path which attains the apparently very loose bound on $B_n$.

Another illustrative example concerns possible degenerate results obtained from deterministic modelling and analysis. This is the case when the arrival process $A(n)$ can take infinitely large values, i.e., $\exists K > 0 \exists \varepsilon_K > 0$ such that $\mathbb{P}(A(n) > K) > \varepsilon_K$ (the Poisson process is an example). For such arrival processes, the regularity constraint from Eq. (20) is only satisfied by the degenerate function $\mathcal{G}(n) = \infty$, which clearly yields the degenerate, and useless, backlog bound $B_n \leq \infty$. However, reiterating the previous argument, this degenerate bound is also tight.

We conclude here by pointing out that if one seeks deterministic bounds from deterministic or even stochastic arrival models, then DNC is an attractive theory (illustrated here by Eqs. (20)-(21)): it has a high modelling potential and it (mostly) yields tight bounds. Otherwise, if one seeks probabilistic bounds, e.g.,

$$\mathbb{P}(B_n > \sigma) \leq \varepsilon(\sigma),$$

where $\varepsilon(\sigma)$ is to be determined, then DNC is an inopportune theory. The main reason is that a purely deterministic analysis can yield extremely loose bounds due to not leveraging from statistical multiplexing gain. This discussion will be continued in Section 5.

4.3 Stochastic Arrival Models

A probabilistic analysis generally requires a probabilistic arrival model. Here we consider three of the main stochastic arrival models proposed in the SNC literature. For the sake of presentation we use the names SBB, SBB, and SBB, and omit original names. The goal of this subsection is to explain their motivation and benefits.

SBB : $\mathbb{P}(A(k, n) - \mathcal{G}(n - k) > \sigma) \leq \varepsilon(\sigma) \forall k, n, \sigma$

SBB : $\mathbb{P}\left( \max_{0 \leq k \leq n} \{ A(k, n) - \mathcal{G}(n - k) \} > \sigma \right) \leq \varepsilon(\sigma) \forall n, \sigma$

SBB : $\mathbb{P}\left( \max_{0 \leq k \leq n} \{ A(k, n) - \mathcal{G}(n - k) \} > \sigma \right) \leq \varepsilon(\sigma) \forall \sigma$

$\mathcal{G}(n)$’s are non-random and are called envelope functions. $\varepsilon(\sigma)$’s are called error functions. Some technical and quite intuitive conditions are that the envelope functions are non-decreasing, whereas the error functions are non-increasing. Note also that a degree of freedom of the bounding approach is that for all three models the arrival process $A(n)$ does not need to be stationary, although the bounds themselves (the envelopes $\mathcal{G}(n)$) are so.

Before we explain the three models, it is important to observe the formation of the bottom two: SBB [17] is formed by inserting the free variable $k$ from SBB (short-hand for stochastically bounded burstiness) [45] into the probability, whereas SBB [26] is further formed by inserting the free variable $n$ as well. Informally, SBB measures events consisting of all past histories of the process $A(n)$, i.e., relative to all times. In turn, SBB measures events consisting of a single past history, i.e., relative to a fixed time, whereas SBB measures events as single fragments of past histories.

Although the SBB model seems the simplest amongst the three, it is actually the SBB model which is the natural
extension of the classic deterministic model from Eq. (20). To see the reason, rewrite Eq. (20) as

$$\max_{0 \leq k \leq n} \{ A(k, n) - G(n - k) \} \leq \sigma \forall n \ .$$ (22)

Note that the $S^2$BB model enforces a bound on the CCDF of the LHS term above. In other words, the $S^2$BB model quantifies, with an upper bound, the probability that the deterministic model is violated by more than $\sigma$. An attractive property of $S^2$BB is that it immediately lends itself to the calculation of performance bounds. Indeed, a straightforward manipulation of Eq. (18) and $S^2$BB yields the bound

$$P(B_n > \sigma_0 + \sigma) \leq \varepsilon(\sigma) \ ,$$ (23)

where $\sigma_0 = \max_{k \geq 0} \{ \hat{G}(k) - Ck \}$ is exactly the deterministic backlog bound from Eq. (21). To recapitulate, $S^2$BB quantifies the violation probabilities of the deterministic model (see $S^2$BB and Eq. (22)), whereas the probabilistic backlog bound quantifies the violation probabilities of the deterministic backlog bound (see Eqs. (23) and (21)). As these violation probabilities are identical, one can argue that $S^2$BB is the 'natural' probabilistic extension of the deterministic arrival model from Eq. (20) (or, equivalently, from Eq. (22)).

In practice, the choice of SBB vs. $S^2$BB depends on the arrivals' input. If the input is a measurement trace, then $S^2$BB should be chosen since it immediately lends itself to performance bounds, as shown earlier. Given a trace $A(n)$ with $n$ elements, an $S^2$BB fitting algorithm would follow the steps: 1) make a guess on $\hat{G}(n)$ (e.g., $\hat{G}(n) = (r + \delta)n$, where $r$ is the average rate of the trace and $\delta > 0$ is a tuning parameter), 2) compute the partial sums $A(n, k)$ and the LHS terms in Eq. (22), and 3) fit a distribution function. Ignoring the accuracy of the fitting, i.e., the range of values $\sigma$, the algorithm runs in $O(n^2)$ time. There is no specific rule for the tuning parameter $\delta$, which is to be optimized numerically.

If the arrivals' input is some random process $A(n)$, then it is generally easier to first fit the SBB model. A typical way is to derive an MGF bound, e.g., $M_{A(n)}(\theta) \leq e^{\theta \sigma}$, for some $\theta > 0$ (see Eq. (16) for Markov arrival processes). If $A(n)$ is also stationary, then an SBB model can be fitted using the Chernoff bound\(^5\), i.e.,

$$P(A(k, n) > r(n - k) + \sigma) \leq e^{-\theta \sigma} \forall k, n, \sigma \ .$$ (24)

This SBB model further lends itself to the $S^2$BB model. Indeed, one can write for some $\sigma > 0$:

$$P(\max_{0 \leq k \leq n} \{ A(k, n) - (r + \delta)(n - k) \} > \sigma) \leq \sum_{\theta \leq k \leq n} P(A(k, n) > (r \delta)(n - k) + \delta k(n - k) + \sigma) \leq \sum_{k \geq 1} e^{-\theta k} e^{-\theta \sigma} \leq \frac{1}{\theta \sigma} e^{-\theta \sigma} .$$

The second line follows from Boole's inequality, whereas the exponential bounds in the last line follow from Eq. (24). What is important to remark is that the transition from the SBB to $S^2$BB involves a rate increase from $r$ to $r + \delta$. This penalty is due to the need to obtain a bounded error function for $S^2$BB; note that, if $\delta$ was zero, then the obtained error function would be unbounded.

\(^5\)For some r.v. $X$ and $x, \theta \in \mathbb{R}_+$, the Chernoff bound is $P(X > x) \leq M_X(\theta) e^{-\theta x}$.

4.4 Quasi-Determinism in the $S^2$BB Model

So far we have only commented on the applicability of the SBB and $S^2$BB models. The reason is that, as we demonstrate in this section, the $S^2$BB model is quasi-deterministic for the class of stationary and ergodic arrival processes. Formally, the quasi-determinism means that the corresponding violation probabilities, set through the error function $\varepsilon(\sigma)$, can only take the extreme values, i.e.,

$$\varepsilon(\sigma) \in \{0, 1\} \forall \sigma \ .$$ (25)

The immediate consequence is that the resulting SNC formulation from [26] is essentially quasi-deterministic, and does not capture statistical multiplexing gain. In fact, multiplexing quasi-deterministic $S^2$BB flows yields quasi-deterministic aggregates, by using the Superposition Property from [26]; for the precise meaning of 'statistical multiplexing gain' we refer to Section 5.

We next prove the quasi-determinism claim for stationary and ergodic processes, and then construct two rather contrived arrival processes for which the $S^2$BB model is not necessarily quasi-deterministic.

4.4.1 Stationary and Ergodic Processes

First we give some definitions (see Breiman [7], pp. 104-120). Consider a random process $X = (X_1, X_2, \ldots)$ defined on some joint probability space $(\Omega, \mathcal{F}, P)$; the Borel $\sigma$-field of the subsets of $\mathbb{R}$ is denoted by $\mathcal{B}$. We denote $I = \{1, 2, \ldots\}$, and the product spaces $\mathbb{R}^I = \{x = (x_1, x_2, \ldots) : x_i \in \mathbb{R}\}$ and $\mathcal{B}^I = \{B = (B_1, B_2, \ldots) : B_i \in \mathcal{B}\}$.

By definition, the process $X$ is (strongly) stationary if

$$P(X_{i_1} \leq x_1, X_{i_2} \leq x_2, \ldots, X_{i_n} \leq x_n) = P(X_{i_1+n} \leq x_1, X_{i_2+n} \leq x_2, \ldots, X_{i_n+n} \leq x_n) ,$$

for all $n, k \in \mathbb{N}_0$, $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n$, and $x_1, x_2, \ldots, x_n$. In other words, stationarity means that the distribution of any sequence $(X_{i_1}, X_{i_2}, \ldots, X_{i_n})$ is invariant under shift.

Further, the notion of ergodicity requires the introduction of an explicit shift operator $T: \mathbb{R}^I \to \mathbb{R}^I$, defined as

$$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots) ,$$

for all sequences $x = (x_1, x_2, \ldots)$. The stationarity of $X$ implies that $T$ is measure preserving, i.e., by definition

$$P(X \in B) = P(TX \in B) \forall B \in \mathcal{B}^I .$$

For $B \in \mathcal{B}$, the event $\{X \in B\}$ is said to be invariant if

$$\{X \in B\} = \{TX \in B\} \text{ P-a.s.} ,$$

i.e., the events $\{X \in B\}$ and $\{TX \in B\}$ differ by a set of probability zero. In other words, the event $\{X \in B\}$ is invariant if its incidence does not depend (a.s.) on any finite prefix of $X$. Finally, the process $X$ is ergodic if any invariant event has probability 0 or 1.

The following lemma will be used to prove the claim of quasi-determinism.

**Lemma 1.** Consider a stationary and ergodic process $X = (X_1, X_2, \ldots)$. Then

$$P(\max \{X_1, X_2, \ldots\} > \sigma) \in \{0, 1\} \forall \sigma \ .$$

The lemma implies that $\max \{X_1, X_2, \ldots\} = K$ a.s., where $K$ is a constant or $K = \infty$. 318
Proof. Fix $\sigma$ and let $B = (-\infty, \sigma]^4$. We shall prove that

$$\{X \in B\} = \left\{ \max_{i \geq 2} X_i \leq \sigma \right\}$$

is an invariant event, which is equivalent to showing that

$$\mathbb{P}(\{X \in B\} \Delta \{TX \in B\} = 0) = 0,$$  \hspace{1cm} (26)

where $\Delta$ denotes the symmetric difference and $\{TX \in B\} = \{\max_{i \geq 2} X_i \leq \sigma\}$.

Let us first note that

$$\mathbb{P}(\max_{i \geq 2} X_i > \sigma) = \lim_{n \to \infty} \mathbb{P}(\max_{i \geq 2} X_i > \sigma) = \lim_{n \to \infty} \mathbb{P}(\max_{i \geq 2} X_{i+n} > \sigma) = \mathbb{P}(\max_{i \geq 2} X_i > \sigma).$$  \hspace{1cm} (27)

The second and last lines follow from the monotone convergence theorem (if $B_n$ is a non-decreasing sequence of events, then $\mathbb{P}(\lim_{n \to \infty} B_n) = \lim_{n \to \infty} \mathbb{P}(B_n)$). The third line follows from the stationarity of $X_k$.

Expanding the symmetric difference from Eq. (26) into the union of two events, we have for the first one

$$\mathbb{P}\left(\bigcap_{i \geq 2} \left(\max_{i \geq 2} X_i \leq \sigma\right) \cup \left(\max_{i \geq 2} X_i > \sigma\right)\right) = \mathbb{P}(\emptyset) = 0.$$

For the second one we use the inclusion-exclusion formula:

$$\mathbb{P}\left(\bigcup_{i \geq 2} \left(\max_{i \geq 2} X_i \leq \sigma\right) \cap \left(\max_{i \geq 2} X_i > \sigma\right)\right) = \mathbb{P}\left(\max_{i \geq 2} X_i > \sigma\right) + \mathbb{P}\left(\max_{i \geq 2} X_i \leq \sigma\right) - 1 = 0.$$

In the last line we applied Eq. (27). Collecting terms implies that Eq. (26) holds and thus the event $\{X \in B\}$ is invariant. Because $X$ is ergodic, it follows that $\mathbb{P}(X \in B) \in \{0, 1\}$, which completes the proof.

We are now ready to demonstrate the quasi-determinism claim. Let the $S^3$BB model from Section 4 for an arrival process $A(n)$, some envelope $G(n)$, and error function $\varepsilon(\sigma)$. Assume that $a_n := A(n - 1, n)$ is stationary and ergodic. It then follows that for any $m \in \mathbb{N}^*$, the block process $\left(X^{(m)}_{n}\right)_{n \geq 2}$ comprising of blocks of $m$ consecutive instances of $a_n$ and defined as

$$X^{(m)}_{n} := A(n, n + m - 1) - G(m) \forall n \geq 1$$

is also stationary and ergodic (cf. [7], Propositions 6.6 and 6.31). According to Lemma 1, there exists $K^{(m)}$ such that

$$\max_{0 \leq k \leq n \leq \infty}(A(k, n) - G(n - k)) = K^{(m)} \ a.s.$$  \hspace{1cm} (28)

for all $m \geq 1$. Taking $K = \max_{m} K^{(m)}$ we obtain that

$$\max_{0 \leq k \leq n \leq \infty}(A(k, n) - G(n - k)) = K \ a.s.$$  \hspace{1cm} (29)

thus concluding that

$$\varepsilon(\sigma) \in \{0, 1\} \ \forall \sigma$$

in the definition of the $S^3$BB model.\footnote{We remark that $S^3$BB is not necessarily quasi-deterministic under the additional assumptions of restricting the ‘max’ operator to a finite interval $0 \leq k \leq n$ and letting $\varepsilon(\sigma)$ depend on the right margin $n$ (see Definition 3.2.1 in [28]).}

### 4.4.2 Non-Stationary or Non-Ergodic Processes

Here we give two examples of arrival processes for which the $S^3$BB model is not necessarily quasi-deterministic. Such processes have to be non-stationary or non-ergodic.

For an example of a non-stationary process consider \( a_1 = r + X, \ a_n = r \ \forall n \geq 2 \), where $X$ is some r.v. satisfying $E[X] > 0$. Note that the cumulative process $A(n) = r n + X \ \forall n$ does not have stationary increments because $E[A_1] \neq E[a_2]$. Moreover, for $\sigma > 0$, the probability

$$\mathbb{P}\left(\max_{0 \leq k \leq n \leq \infty}(A(k, n) - r(n - k)) > \sigma\right) = \mathbb{P}[X_\sigma > \sigma]$$

can be different from zero and one.

For an example of a non-ergodic process consider

$$a_n = X \ \forall n \geq 1,$$

for some r.v. $X$. The cumulative process $A(n) = nX$ is stationary but not ergodic, as there are many realizations of the process for which the time averages are different. To construct a non-quasi-deterministic $S^3$BB model, one can take $G(n) = r n$ and $X$ be any Bernoulli r.v. with $E[X] = r$ but $\mathbb{P}(X \neq r) > 0$.

What the two examples have in common is that the sample-paths are completely determined from some time scale on. In particular, in the second example, the sample-paths are completely determined once time starts. We speculate that more compounded examples would also account for randomness but for a finite time scale only, in order to avoid the limiting argument in the preceding quasi-determinism proof. Due to this rather unnatural restricted capability in capturing randomness, the relevance of such models is unclear.

### 5. STATISTICAL MULTIPLICATING GAIN

In this section we justify the raison d’être of SNC; concretely, we present a result which rigorously reveals the magnitude of the statistical multiplexing gain, as a scaling law, achieved by SNC in the single-node case. Then we discuss on the existence of multiplexing gain in the multi-node case, and present numerical results.

#### 5.1 Single-Node Case

Statistical multiplexing is an essential property of packet-switched networks, which are based on the principle of resource sharing. It basically says that the number of resources needed to support service for (say $N$) flows is much smaller than $N$ times the number of resources needed to support service for a single flow. The raison d’être of SNC is to capture the gap between these two quantities, i.e., the statistical multiplexing gain, while closely reproducing the elegant methodology of DNC.

To illustrate the magnitude of the statistical multiplexing gain achieved with SNC we consider a node of capacity $C$ serving $N$ flows $A_j(n)$, each modelled with the envelope

$$A_j(k, n) \leq r(n - k) + b \ \forall k, n,$$  \hspace{1cm} (28)

where $r > 0$ is a rate and $b \geq r$ is a burst size. Consider now the design question Q1: “How large should $C$ be such that the delay is smaller than some value, normalized to 1?” To answer, it is convenient to derive a backlog bound. Assuming the stability condition $\rho := \frac{a_n}{\mathbb{E}[a_n]} \leq 1$, the bound follows
directly from Eq. (21), where \( G(k) = Nrk \):
\[
B_n \leq Nb \cdot \varepsilon \cdot n \cdot \varepsilon .
\]
Because of the delay normalization to 1, which implies that \( C \) and the backlog scale identically, we conclude that the required capacity scales as \( C = \mathcal{O}(N) \) in the burst \( b \). Although this conclusion is based on a tight bound (recall the discussion from Section 4.2), the intuition is that a much smaller capacity would be sufficient under broad statistical assumptions on the flows \( A_j(n) \), and as long as some violations of the delay constraint are tolerable.

Let us additionally assume that \( A_j(n) \)'s are stationary and statistically independent, and enforce the (tolerable) constraint \( P(\text{delay} > 1) \leq \varepsilon \), where \( \varepsilon \) is some small value, e.g., \( \varepsilon = 10^{-3} \). With these assumptions, we can use a stochastic bound on the aggregate \( A(n) := \sum_{j=1}^{N} A_j(n) \), i.e., [38]
\[
P(\text{delay} > 1) \leq e^{\frac{-b^2}{2\varepsilon}} \forall \varepsilon \geq 0 . \quad (29)
\]
A backlog bound can then be computed as in Eq. (19):
\[
P(B_n > \sigma) \leq \frac{P(\text{delay} > 1)}{\sigma^2} \leq \frac{\sum_{k=1}^{N} P(A(k) > Nrk + (C - Nrk)k + \sigma)}{\int_{0}^{\infty} e^{-\frac{1}{2\varepsilon}((Nrk+\sigma) - \sigma^2)/(\varepsilon^2/2\varepsilon^2)} ds .}
\]
In the last line we used the bound from Eq. (29) and bounded a sum of non-increasing terms by an integral. With the change of variable \( u = \frac{(Nrk+\sigma) - \sigma^2}{\varepsilon^2/2\varepsilon^2} \) the last term becomes
\[
\frac{\sqrt{Nb}}{C - Nr} \int_{0}^{\infty} e^{\frac{-u^2}{2\varepsilon^2}} du \leq \frac{b^2}{\varepsilon^2} \cdot e^{\frac{-2\varepsilon^2}{2\varepsilon^2}} .
\]
Here we used Gordon’s inequality for the standard normal density function, i.e., \( \int_{0}^{\infty} e^{-\frac{x^2}{2\varepsilon^2}} dx \leq \frac{\sqrt{\pi}}{\varepsilon} \cdot e^{\frac{-2\varepsilon^2}{2\varepsilon^2}} \). Setting the last term to \( \varepsilon \) yields
\[
\sigma^2 = 2Nb^2 \left( \log \frac{b^2}{\varepsilon^2} \cdot e^{\frac{-2\varepsilon^2}{2\varepsilon^2}} - \log \varepsilon \right) .
\]
From here it follows that \( C = \mathcal{O} \left( \sqrt{N} \right) \) in the burst \( b \) (recall that \( C \) scales identically with the backlog \( \sigma \)); note, however, that \( C = \mathcal{O}(N) \) in the rate \( r \) in order to satisfy the stability condition. The \( \mathcal{O} \left( \sqrt{N} \right) \) law can also be deduced from a result from [4], which is however obtained using an approximative application of the Central Limit Theorem, and hence not rigorous. Several other probabilistic bounds on multiplexed deterministically regulated arrivals (as in Eq. (28)) exist in the literature, e.g., [47, 50]; however, they do not appear to easily lend themselves to the \( \mathcal{O} \left( \sqrt{N} \right) \) law.

The difference in the scaling laws \( C = \mathcal{O}(N) \) (obtained with DNC) vs. \( C = \mathcal{O}(\sqrt{N}) \) (obtained with SNC) reveals thus the magnitude of the statistical multiplexing gain achieved with SNC as \( \mathcal{O} \left( \sqrt{N} \right) \).

5.2 Multi-Node Case

Lastly, we discuss on an unconventional type of statistical multiplexing which, to the best of our knowledge, has not been raised previously. Consider the tandem network from Figure 2 in which a flow \( A \) crosses \( M \) nodes in series; at each node \( j = 1, \ldots, M \) along the end-to-end (e2e) path, \( A \) shares the local resource (the capacity \( C \)) with a local cross flow \( A_j \). All flows are stationary and statistically independent. This type of resource sharing looks similar to the conventional one, except that the "resource" is now a distributed one (i.e., all the capacities) and the cross flows do not share the same resource with each other. The arising question concerns the existence of a distributed multiplexing gain.

To answer, we apply and compare DNC and SNC for the following scenario: \( A \) is bounded by the envelope from Eq. (28) with rate \( r \) and burst \( b \), and \( A_j \)'s are bounded by the same envelope but with rate \( Nr \) and burst \( Nb \) (\( N \) will be used for tuning conventional (per-node) multiplexing gain).

We enforce the stability condition \( C \geq (N + 1)r \) and assume that flow \( A \) gets lower priority at each node. We ask the design question Q2: "How large should \( C \) be such that the e2e delay of \( A \) is smaller than 1?".

To deal with the additional complexities due to scheduling and multi-node (i.e., the system’s ‘noise’), we run the network calculus engine, i.e., transform the network system into a ‘somewhat looking’ linear system. The first step is to derive the service processes at each node, i.e., \( S(n) = \lfloor (C - Nr)n - Nb \rfloor \), and then apply the convolution formula from Eq. (13) yielding \( S(n) = \lfloor (n - Nb) \rfloor (C - Nr) \).

From the transformed system, consisting of the input \( A \) and the service process \( S \), the deterministic e2e delay bound is \( W \leq \mathcal{O}(N + 1, M) \). A probabilistic e2e delay bound can be also derived (not shown here) using the SNC formulation from Fidler [22] and the representation from Eq. (29).

Figure 3 illustrates the required capacities \( C \), computed with DNC and SNC, for the questions Q1 in (a) and Q2 in (b). Note that (a) clearly shows the (conventional) multiplexing gain when \( b = 3r \). Note also that the multiplexing gain kicks-in around ten flows. In turn, (b) illustrates that there is no distributed multiplexing gain when there is only one cross flow \( N = 1 \); however, for sufficiently large \( N \), e.g., 50, the conventional multiplexing gain kicks-in and compensates for the lack of distributed multiplexing gain.

Figure 2: A tandem network with cross traffic

Figure 3: Capacity dimensioning, with DNC and SNC, for the delay constraints (delay \( \leq 1 \)) and \( P(\text{delay} > 1) \leq \varepsilon \), respectively (\( r = 1, \varepsilon = 10^{-3} \))
6. CONCLUSIONS

Network calculus (NC) has seen a lot of research over the last 20 years, e.g., Google Scholar yields ≈ 3000 hits when searching for “network calculus”, comparing relatively well to its ‘older brother’ queueing theory, which yields ≈ 60000 hits. The broad perception on NC is that it holds good promises as an alternative/complementary methodology to classical queueing theory. Yet, in the camp of NC researchers as well as the larger audience, the mood somewhat oscillates between “Hooray, we found the Holy Grail!” and “Oh no, it’s not gonna work.” In this paper, we have tried to better locate where the truth stands and aimed at reducing the level of confusion—and hopefully not creating new one—by clarifying some important issues related to the core modelling abstractions in network calculus: arrival envelopes and service processes. On this mission, we also provided some new insights into network calculus in the larger frame of things. Specifically, we collected the following facts:

• NC is about approximating a complex (typically nonlinear) queueing system by a min-plus linear one. The approximation closely follows the traces of both traditional LTI system theory and of elementary properties from classical queueing theory; the analogy is good, but not perfect.

• With respect to arrival envelopes we provided a number of pitfalls as well as advice on how to avoid them. The most salient and profound observation is that the SBB envelope model presented at ACM Sigcomm 2006 delivers quasi-deterministic performance bounds for a large class of arrivals (stationary and ergodic), which means that it cannot generally capture statistical multiplexing gain.

• Statistical multiplexing gain can be captured well by SNC, under carefully defined envelope and service processes. In particular, for the application scenario of multiplexing independent regulated arrival processes, we rigorously showed the gain to be on the order of Ω(√N), which before had only been shown approximately. So there is light at the end of the tunnel and hopefully it is not by a train railing towards us.

• We have also discussed on the tightness of bounds, a lingering issue surrounding almost any discussion on NC. We clarified issues about the tightness of the DNC and provided insights into that of SNC. In short, DNC generally delivers tight bounds, i.e., the bounds can be attained; SNC is as tight as the underlying probability inequalities being used (Boole, Chernoff, martingale inequalities). As these inequalities can be plugged into SNC in a modular fashion, one may argue that the SNC analysis can be made as tight as the state-of-the-art in probability theory allows.

Summing up, while using NC does not provide us with a free lunch, it still seems to be of good value in analyzing traditionally hard fundamental queueing problems due to its scalable tradeoff between accuracy and ease of analysis.

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8. REFERENCES


