Distributed Computer Systems Lab
http://disco.informatik.uni-kl.de

Performance Modelling of Distributed Systems

3. Modelling of the Arrival Processes

Prof. Dr.-Ing. Jens B. Schmitt
(jschmitt@cs.uni-kl.de)

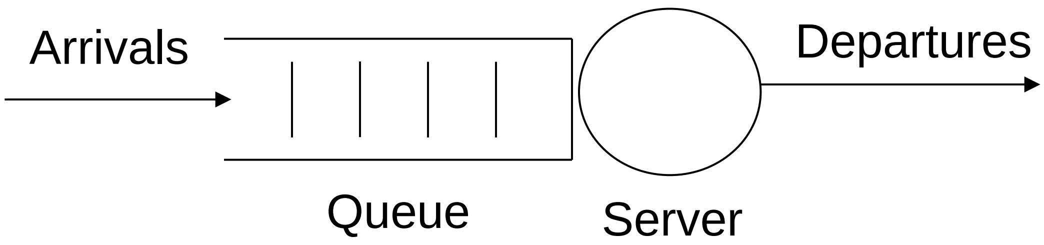
Performance Analysis of Distributed Systems

- Classical method: Queueing Theory (QT)
  - Huge success: telephone network
  - Poisson arrivals: M/M/1, etc.
  - Product-form networks rely on Poisson assumption
  - Mainly for **average-case** analysis

- Lately: Deterministic Network Calculus (DNC)
  - Get rid of stochastic assumptions
  - Work out **worst-case** behaviour
  - Elegant network analysis without many assumptions
  - Yet, does not capture statistical multiplexing
- Most recently: Stochastic Network Calculus (SNC)
  - Falls in the middle between QT and DNC: **probabilistic worst-case**
  - Captures statistical multiplexing
  - Promises elegant network analysis

The „Canonical“ Problem

Goal: Compute delay for a single flow at a single server



- QT approach: knowledge about arrival and service distributions
- DNC approach: deterministic bounds on arrivals and service
- SNC approach: **probabilistic** bounds on **arrivals** and service

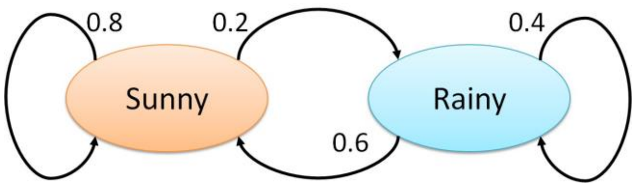
Probabilistic Bounds on Arrivals: Preliminaries

- Several ways to do it, two mainstreams
  - MGF bounds
  - Tail bounds

- Cumulative functions $A(n)$
  - Here: discrete time mainly
  - Deterministic arrival curve α

$$\forall m \leq n : A(n) - A(m) \leq \alpha(n-m)$$

- Background: Stochastic Processes
  - Trajectory / Sample Path
  - Example: Markov chain
  - Here: time space is typically $\mathbb{N}_0$, with the state space being $\mathbb{R}_0^+$
  - Increments of a stochastic process



Why Deterministic Bounds Do not Work?

- Bernoulli process

- Exponentially distributed increments

- Bottom line: best possible arrival curve is bad - at best.

Probabilistic Bounds on Arrivals: Tail Bound

- What we want is a probabilistic extension of the arrival curve

$$\forall m \leq n : \mathbb{P}(A(n) - A(m) \leq \alpha(n-m,\varepsilon)) \geq 1-\varepsilon$$

- Or, equivalently

$$\forall m \leq n : \mathbb{P}(A(n) - A(m) > \alpha(n-m,\varepsilon)) < \varepsilon$$

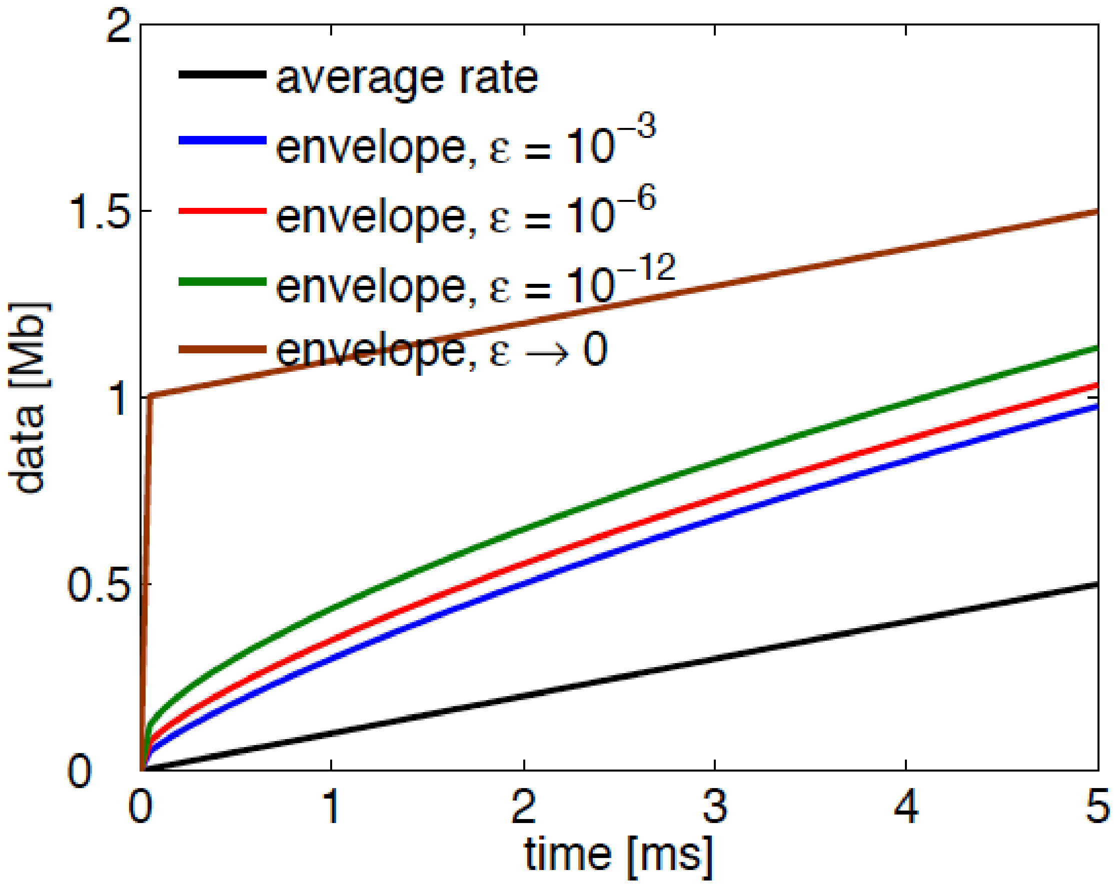
- Definition: a flow A is tail-bounded by envelope α with error η, if for all $\varepsilon > 0$ and $n \geq m \in \mathbb{N}_0$ it holds that:

$$\mathbb{P}(A(n) - A(m) > \alpha(n-m,\varepsilon)) \leq \eta(n-m,\varepsilon)$$

- Example: Exponentially Bounded Burstiness [YaronSidi93]

Probabilistic Bounds on Arrivals: Potential Gain

Envelopes of $n = 100$ multiplexed leaky bucket sources with $\sigma = 10$ kb and $\rho = 1$ Mbps.



Probabilistic Bounds on Arrivals: MGF Bound

- Some inequalities first
  - Markov‘s Inequality

$$\mathbb{P}(X > x) \leq \frac{\mathbb{E}(X)}{x}$$

  - Chernoff‘s Inequality

$$\mathbb{P}(X > x) \leq e^{-\theta x}\phi_X(\theta)$$

- Definition: A flow A is $(\sigma(\theta), \rho(\theta))$-bounded for some $\theta > 0$, if for all $n \geq m \geq 0$ the MGF $\phi_{A(n)-A(m)}(\theta)$ exists and $\phi_{A(n)-A(m)}(\theta) \leq e^{\theta\rho(\theta)(n-m)+\theta\sigma(\theta)}$ holds.
- Instead of a linear (MGF) envelope a general function $f(\theta, n-m)$ can also be used.

Tail Bound vs. MGF Bound

- Conversion Theorem:

  Assume a flow A. If A is tail-bounded by envelope α with error $\eta(k, \varepsilon) = \varepsilon$, it is also $f(\theta, \cdot)$-bounded with:

  $$f(\theta, \cdot) := \int_0^1 e^{\theta\alpha(n-m,\varepsilon)} d\varepsilon.$$

  Conversely if A is $f(\theta, \cdot)$-bounded, it is also tail-bounded with $\alpha(n-m, \varepsilon) = \varepsilon$ and

  $$\eta(n-m, \varepsilon) = f(\theta, n-m)e^{-\theta\varepsilon}.$$

- Note: conversion may involve loss of accuracy.

MGF and Tail Bound: Examples

- Exponentially distributed increments
  - MGF and tail bound exist
  - Use conversion theorem to go from MGF to tail bound

- Bernoulli arrival process
  - Note that a linear MGF bound does not exist
  - Tail bound can again be obtained using Conversion Theorem

- I.i.d. increments with bounded variance
  - E.g. log-normal distribution, has moments but no MGF
  - Use of Chebyshev Inequality

$$\mathbb{P}(X - \mathbb{E}(x) > \varepsilon) \leq \varepsilon^{-2} Var(X)$$

MGF Bound: Multiplexing

- Lemma (Independent Flow Multiplexing)

Let A and B be two flows, which are $(\sigma_A(\theta), \rho_A(\theta))-$ and $(\sigma_B(\theta), \rho_B(\theta))$ – bounded, respectively (for the same $\theta > 0$). Further let A and B be stochastically independent. We call $A \oplus B(n) := A(n) + B(n)$ the multiplexed flow. We have then that $A \oplus B$ is bounded by $(\sigma_A(\theta) + \sigma_B(\theta), \rho_A(\theta) + \rho_B(\theta))$.

- Dependent Case
  - Use of Hölder‘s Inequality

$$\mathbb{E}(XY) = (\mathbb{E}(X^p))^{\frac{1}{p}}(\mathbb{E}(Y^q))^{\frac{1}{q}}$$