Excercise Sheet 1

Performance Modeling of Distributed Systems

Winter Term 2013/2014

Assignment 1 (Discrete Probability Spaces)
Assume a discrete probabillity space, with $\Omega = \{\omega_1, \omega_2, \ldots\}$ and p_i being probability weights. Show that the function $\mathbb{P}$ defined as:

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} p_i \delta_A(\omega_i)$$

is a probability measure. Show further that the probability measure is uniquely defined, i.e. assume another function $\mathbb{P}'$ with $\mathbb{P}(\{\omega_i\}) = \mathbb{P}'(\{\omega_i\})$ for all $i \in \mathbb{N}$ and show:

$$\mathbb{P}'(A) = \mathbb{P}(A) \qquad \forall A \in \mathcal{A}$$

Assignment 2 (Probabilities)
Let $(\Omega, \mathcal{A})$ be a measurable space and $A, B \in \mathcal{A}$. Prove in this sequence:

- $\mathbb{P}(A) \geq \mathbb{P}(B)$ if $A \supseteq B$
- $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$
- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- $\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B)$

Assignment 3 (Generating Random Variables)
Many programming languages offer methods to create random numbers in $[0, 1]$. How can you use this to create realisations of an exponential distributed random variable with parameter λ? Use this method to simulate n different exponential distributed random variables X_i with the same parameter λ and compute the arithmetic mean of these realisations $E = \frac{1}{n}\sum_{i=1}^{n} X_i$. How does $E - \frac{1}{\lambda}$ evolve for large n?

Use this procedure to generate exponential distributed random numbers $x_1, x_2, x_3, \ldots$ with parameter λ. Now count the occurences of this random numbers in certain intervals. Define these intervals of length $N \in \mathbb{R}$ by:

$$I_n = [n \cdot N, (n+1) \cdot N) \subset \mathbb{R} \qquad n \in \mathbb{N}_0$$

The number of realisations in one interval is then given by:

$$f^*(n) = \sum_{i=1} \delta_{I_n}(x_i)$$

Plot the sequence $(f^*(n))_{n \in \mathbb{N}}$. Compare this sequence with the density of an exponentially distributed random variable X with the same parameter λ.

Assignment 4 (Geometric Distributions)
A geometric distributed random variable X is a discrete random variable with probability weights:

$$\mathbb{P}(X = n) = (1-p)^n p \qquad n \in \mathbb{N}_0$$

for some parameter $p \in (0,1)$. Think about a method to generate geometric distributed random numbers.

(Hint: $\mathbb{P}(X = n) = (1-p)^n p = (1-p) \cdot (1-p)^{n-1} p = (1-p) \cdot \mathbb{P}(X = n-1)$)

Generate some geometrically distributed random variables and plot the result.

Take the sequence $(f^*(n))_{n \in \mathbb{N}}$ of the previous exercise, with parameter $\lambda = -\log(1-p)$ and intervals $I_n = [n, n+1)$ and compare it with the plot of geometrically distributed random variables.

Prove that for X being an exponentially distributed random variable with parameter $\lambda = -\log(1-p)$ holds

$$\mathbb{P}(X \in [n, n+1)) = \mathbb{P}(Y = n)$$

where Y is a geometrically distributed random variable with parameter p.