Excercise Sheet 2

Performance Modeling of Distributed Systems

Winter Term 2013/2014

Assignment 1 (Correlation and Dependence)
Let X and Y be stochastically independent and Bernoulli-distrubuted with the same parameter p (this means $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$). Show that the two new random variables $X + Y$ and $X - Y$ are uncorrelated, yet not independent. (Hint: Use the properties of Covariance and Expectations as presented in the lecture or in chapter 1.2 of the script.)

Assignment 2 (Law of total probability)
Asssume some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and some $A \in \mathcal{A}$. Show the law of total probability:

$$\mathbb{P}(A) = \sum_{n=1}^{\infty} \mathbb{P}(A|B_n)\mathbb{P}(B^c)$$

Here the sets B_n are a partition of the set Ω meaning: $B_i \cap B_j = \emptyset$ for all i, j and $\bigcup_{n=1}^{\infty} B_n = \Omega$.

Assignment 3 (MGF and Exponential Distributions)
The above law can be generalized to distributions with density. For that assume two random variables X, Y with densities f_X, f_Y. It holds then:

$$\mathbb{P}(X \in A) = \int_{-\infty}^{\infty} \mathbb{P}(X \in A|Y = y) f_Y(y) dy$$

Assume now X to be exponentially distributed with parameter λ and Y being stochastically independent of X. Use the above generalized law of total probability and the fact that $\mathbb{E}(h(Y)) = \int_{-\infty}^{\infty} h(y) f_Y(y) dy$ to show:

$$\mathbb{P}(X > Y) = \mathbb{E}(e^{-\lambda Y})$$

Assignment 4 (MGF and Moments)
Let X be some real random variable, such that the MGF of X exists in an interval around 0. Calculate the first derivative of the MGF at zero $\frac{d}{d\theta}\phi_X(\theta)\big|_{\theta=0}$. Calculate the n-th derivative of the MGF at zero $\left(\frac{d}{d\theta}\right)^n \phi_X(\theta)\big|_{\theta=0}$. (Hint: Use that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ and use $\mathbb{E}(\sum_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} \mathbb{E}(X_n)$.)

Assignment 5 (Log-Normal Distribution)
The log-normal distribution is defined by the following density:

$$f(x, \mu, \sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\left(\frac{(\ln x - \mu)^2}{2\sigma^2}\right)} \qquad \forall\, x > 0$$

Its moments are given by $\mathbb{E}(X^n) = e^{n\mu + \frac{1}{2}\sigma^2 n^2}$. Show that for all $\theta > 0$ the corresponding MGF does not exist.