Excercise Sheet 5

Performance Modeling of Distributed Systems

Winter Term 2013/2014

Assignment 1 (Server Concatenation I)
Assume a dynamic S- and a dynamic T-server. Assume them both being $\sigma(\theta), \rho(\theta)$-bounded and that the departures of the dynamic S-server are the arrivals for the dynamic T-server. Show that for $\rho_S(\theta) = \rho_T(\theta)$ the dynamic $S \otimes T$-server is $(\sigma(\theta), \rho(\theta))$-bounded with:

$$\rho(\theta) := \rho_S(\theta) + \frac{1}{\theta}$$

and

$$\sigma(\theta) := \sigma_S(\theta) + \sigma_T(\theta)$$

(Hint: Use the simple inequality $n + 1 \leq e^n$ for all $n \in \mathbb{N}_0$)

Assignment 2 (Server Concatenation II)
Let a dynamic S-server be $(\sigma(\theta), \rho(\theta))$-bounded. Consider the following expression:

$$s_S^*(\theta) = \limsup_{m \to \infty} -\frac{1}{\theta m} \log \mathbb{E}(e^{-\theta S(n,n+m)})$$

Show that $s^*(\theta) \leq -\rho(\theta)$. Assume now the situation as in theorem **??**. Show that for the concatenated server holds:

$$s_{S \otimes T}^*(\theta) \leq -\rho_T(\theta)$$

for all $\rho_T(\theta)$ with $|\rho_T(\theta)| < |\rho_S(\theta)|$. Next show that also for the case $\rho_T(\theta) = \rho_S(\theta)$ holds:

$$s_{S \otimes T}^*(\theta) \leq -\rho_T(\theta)$$

Assignment 3 (Conversion of Deconvolution)
Let $X, Y \in \mathcal{S}$ be two stochastically independent random processes. Prove:

$$\phi_{X \oslash Y(m,n)}(\theta) \leq (\phi_X(\theta) \circ \phi_Y(-\theta))(m, n)$$

for all $\theta > 0$ and $(m, n) \in \Lambda(\mathbb{N}_0)$ such that the above MGFs exist.

Assignment 4 (Multiplexed Flows)
We want to investigate how a system with increasing number of flows scales. Assume the following scenario: We have one node with a constant service rate $c = 1$ (i.e. $S(m, n) = c(n - m)$ for all $m \leq n \in \mathbb{N}_0$) and N stochastically independent flows arrive at this node. All of the flows share the same distribution for their increments a_i ($i = 1, \ldots, N$). All increments are i.i.d., we consider three different cases of what this distribution looks like:

- $a_i(m) = 1$ with probability $\frac{1}{N}$ and $a_i(m) = 0$ with probability $1 - \frac{1}{N}$
- $a_i(m) = \frac{2}{N}$ with probability $\frac{1}{2}$ and $a_i(m) = 0$ with probability $a_i(m) = \frac{1}{2}$
- $a_i(m) = N$ with probability $\frac{1}{N^2}$ and $a_i(m) = 0$ with probability $1 - \frac{1}{N^2}$

Give for each case the corresponding $f(\theta, n)$-bound and calculate the backlog bound at time $n = 1$ for a fixed N. How do these bounds evolve for large N? What is

$$\lim_{N\to\infty} \mathbb{P}(q(1) > 2)$$

for each case? Why do they behave differently? Analyse the variance of the increments!

(Hints: For the first case you need that $(1 + \frac{t}{N})^N \xrightarrow{N\to\infty} e^t$, for the second case that $(\frac{1}{2}(e^{\frac{2\theta}{N}} + 1))^N \xrightarrow{N\to\infty} e^\theta$. The variance of a binomial distribution with parameters N and p is given by $\mathrm{Var}(B) = Np - Np^2$)